Low Energy Skyrmion-Skyrmion Scattering

T. Gisiger and M. B. Paranjape

Laboratoire de Physique Nucléaire, Université de Montréal, C.P. centreville
Montréal, Québec, Canada H3C 3J7

ABSTRACT

We study the scattering of Skyrmions at low energy and large separation using the method proposed by Manton of truncation to a finite number of degrees freedom. We calculate the induced metric on the manifold of the union of gradient flow curves, which for large separation, to first non-trivial order is parametrized by the variables of the product ansatz.

The calculation of the scattering of baryons is an intractable problem in Q.C.D.. Some progress can be made using the low-energy effective, field theoretic description. The Skyrme model[4] corresponds to the effective degrees of freedom of low energy Q.C.D.. The Skyrme model is described by the Lagrangean,

$$\mathcal{L} = \frac{f_\pi^2}{4} tr(U^\dagger \partial_\mu U U^\dagger \partial^\mu U) + \frac{1}{32e^2} tr([U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2)$$

where $U(x)$ is a unitary matrix valued field. We take

$$U(x) \in SU(2).$$

The Skyrme Lagrangean contains the first terms of a systematic expansion in derivatives of the effective Lagrangean describing low energy interaction of pions. It is derivable from QCD hence f_π and e are in principle calculable parameters. It includes spontaneous breaking of chiral symmetry

$$\begin{aligned}
&SU_L(2) \times SU_R(2) \rightarrow SU_V(2) \quad \text{with} \\
&U(x) = 1 + i\vec{\pi}(x) \cdot \vec{\tau} \cdots \quad \text{where} \\
&\vec{\pi}(x) \sim \quad \text{pions.}
\end{aligned}$$

What is even more surprising is that it includes the baryons as well. They arise as topologically stable, solitonic solutions of the equations of motion. The original proposal of this by Skyrme [4] in the 60's was put on solid

footing by Witten [5] in the 80's. The solitons, which are called Skyrmions, correspond to non-trivial mappings of $\mathbb{R}^3$ plus the point at infinity into $SU(2)$:

$$U(x) : \mathbb{R}^3 + \infty \to SU(2) = S^3.$$

But

$$\mathbb{R}^3 + \infty = S^3$$

thus the homotopy classes of mappings

$$U(x) : S^3 \to S^3$$

which define

$$\Pi_3(S^3) = \mathbb{Z}$$

characterize the space of configurations. The topological charge of each sector is given by

$$N = \frac{1}{24\pi^2} \int d^3\vec{x}\, \epsilon^{ijk} tr(U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U)$$

which is identified with the baryon number. The Skyrmion solution is given by the configuration

$$U_S(\vec{x}) = e^{if(|\vec{x}|)\hat{x}\cdot\vec{\tau}}$$

where $f(|\vec{x}|)$ is a decreasing function which starts at π at the origin and achieves 0 at $|\vec{x}| = \infty$, asymptotically varying as $\frac{\kappa}{|\vec{x}|^2}$. A Skyrmion at position $\vec{R}$ with orientation $A \in SU(2)$ corresponds to the configuration

$$U(A, \vec{R}, \vec{x}) = AU_S(\vec{x} - \vec{R})A^\dagger.$$

Quantum states of definite momentum, isospin and spin obtained by quantizing $\vec{R}$ and A correspond to the baryons; the nucleons, deltas, etc..

$$\vec{J} = \vec{L} + \vec{T}$$

is conserved, however,

$$\left(\vec{L}\right)^2 = \left(\vec{T}\right)^2.$$

The nucleons correspond to

$$\left|\vec{L}\right| = \left|\vec{T}\right| = \frac{1}{2}.$$

With the input of two parameters, f_π and e, everything else can be predicted for the baryons. Agreement with experiment is within $10\% \sim 30\%$ for $M_N, M_\Delta - M_N, < r^2 >^{\frac{1}{2}}_{T=0,1}, \mu_p, \mu_n, g_A, g_{\pi NN} \cdots$[10].

The sector with $B = 2$, should contain the deuteron as a bound state of minimal energy and the scattering of two nucleons. Even the classical scattering is too difficult to compute. There are an infinite number of degrees of freedom. One has to in principle solve a non-linear partial differential equation of motion for the time evolution.

An idea put forward by Manton[1] was to look for an appropriate truncation of the degrees of freedom. He first considers the case of theories of the Bogomolnyi type, those theories which admit *static* soliton solutions, usually in the topological two soliton sector, which asymptotically describe two single solitons at arbitrary positions and relative orientations. The configuration at small separation contains, in general, strong deformations of the individual solitons and in fact they lose their identity. The set of configurations have, however, the same energy since they correspond to the continuous variation of a finite number of parameters, the modulii. Otherwise they could not be stationary points of the potential. In general, for solitons corresponding to a topological quantum number, the modulii space corresponds to the sub-manifold of minimum energy configurations within the given topological sector. Manton suggests that the low energy scattering of solitons, with initial configuration on this sub-manifold corresponding to asymptotic, single solitons, with arbitrarily small initial velocity tangent to the sub-manifold, will self-consistently be constrained to remain on the sub-manifold. Since the potential energy is a constant on the sub-manifold the resulting dynamics reduces to geodesic motion on the sub-manifold in the induced metric on the sub-manifold from the kinetic term. It is a difficult task to prove such a truncation of degrees of freedom in a mathematically rigorous fashion, however, it does seem intuitively correct. The non-linearity of the theory implies the coupling of the degrees of freedom corresponding to the sub-manifold with all other excitations through the potential. We are assuming that these are negligible. Manton and Gibbons [2] applied this program with remarkable success to the case of magnetic monopoles in the BPS limit and it has also been applied to vortex scattering in a similar limit [3].

The generalization to the more common situation where the set of static solutions correspond to a finite set of critical points proceeds as follows. The critical points are typically a minimum energy configuration which is essentially a bound state of two solitons, an asymptotic critical point which corresponds to two infinitely separated solitons and possibly a number of unstable non-minimal critical points of varying energies of the same order. These critical points are degenerate with a finite number of degrees of freedom. They are connected by special paths, the paths of steepest descent or equivalently the gradient flow curves. In this case Manton proposes that the dynamics will be constrained to lie on the sub-manifold comprising of the union of all these curves. This again is intuitively reasonable. If we think of the space of all configurations as a large bag, the bottom surface of the bag will correspond

to this sub-manifold, and a slow moving marble rolling on the bottom will tend to stay there.

The Skyrme model falls into the second case. We identify the corresponding sub-manifold for well-separated Skyrmions and we calculate the induced metric to lowest non-trivial inverse order in the separation from the kinetic term. This is the first step towards calculating the scattering of Skyrmions in this formalism.

Thus for the scattering of two Skyrmions, we are looking at the sector of baryon number equal to 2. In this sector the minimum energy configuration should correspond to the bound state of two Skyrmions, which must represent the deuteron. The asymptotic critical point corresponds to two infinitely separated Skyrmions. There exist, known, non-minimal critical points, corresponding to a spherically symmetric configuration, the di-baryon solution [6]. The energy of this configuration is about three times the energy of a single Skyrmion. There are also, possibly, other non-minimal critical points with energy less than two infinitely separated Skyrmions [7]. The scattering of two Skyrmions will take place on the union of the paths of steepest descent which connect the various critical points.

We consider the scattering only for large separation. In this way we do not have to know the structure of this manifold in the complicated region where the two Skyrmions interact strongly and consequently are much deformed. In the region of large separation the product ansatz corresponds to

$$\begin{aligned} U(\vec{x}) &= U_1(\vec{x}-\vec{R}_1)U_2(\vec{x}-\vec{R}_2) \\ &= AU(\vec{x}-\vec{R}_1)A^\dagger BU(\vec{x}-\vec{R}_2)B^\dagger \end{aligned}$$

where $U(\vec{x}-\vec{R}_1)$ and $U(\vec{x}-\vec{R}_2)$ correspond to the field of a single Skyrmion solution centered at R_1 and R_2 respectively. The full Skyrme model dynamics implies a deformation of each Skyrmion. We will neglect this deformation.

It remains to calculate the metric on the sub-manifold parametrized by the product ansatz. We replace $\vec{R}_1-\vec{R}_2$ by $\vec{d}$ placing us in the center of mass reference frame and reducing the number of degrees of freedom of the system to nine. We find, along with Schroers[9] the interesting result that the metric behaves like $1/d$ where $d=|\vec{d}|$ is the separation. We find the kinetic energy:

$$\begin{aligned} T = &-2M + \frac{1}{4}M\dot{\vec{d}}^{\,2} + 2\Lambda\big(\mathcal{L}^a(A)\,\mathcal{L}^a(A) + \mathcal{L}^a(B)\,\mathcal{L}^a(B)\big) \\ &+ \frac{\Delta}{d}\epsilon^{iac}\epsilon^{jbd}\,\mathcal{R}^c(A)\,\mathcal{R}^d(B)\,\left(\delta^{ij}-\hat{d}^i\hat{d}^j\right)D_{ab}(A^\dagger B) + O(1/d^2). \end{aligned}$$

where

$$M = 4\pi \int_0^\infty r^2 dr$$
$$\times \left\{ \frac{1}{8} f_\pi^2 \left[\left(\frac{\partial F}{\partial r} \right)^2 + 2 \frac{\sin^2 F}{r^2} \right] + \frac{1}{2e^2} \frac{\sin^2 F}{r^2} \left[\frac{\sin^2 F}{r^2} + 2 \left(\frac{\partial F}{\partial r} \right)^2 \right] \right\}$$

is the mass of a Skyrmion and

$$\Lambda = (e f_\pi)^3 \int r^2 dr \sin^2 F \left[1 + \frac{4}{(e f_\pi)^2} \left(F'^2 + \frac{\sin^2 F}{r^2} \right) \right]$$

is its inertia momentum and where $\Delta = 2\pi\kappa^2 f_\pi^2$, $F(r) \sim \kappa/r^2$ at large r, $\mathcal{R}^a(A) \equiv \mathcal{R}_0^a(A)$ and $\hat{d} = \vec{d}/d$. The metric can be easily obtained from this expression by choosing local coordinates on the product ansatz manifold and extracting the quadratic form relating their time derivatives.

The potential [7] between two Skyrmions can be calculated to give

$$V = 2\Delta \frac{(1 - \cos\theta)(3\,(\hat{n} \cdot \hat{d})^2 - 1)}{d^3}$$

where θ, $\hat{n}$ pick out the element of $SU(2)$ given by $A^\dagger B$. The potential is clearly of higher order than the metric, hence the dominant contribution to the scattering at large separation comes only from the metric. Thus to leading order we may even neglect the potential and then the problem reduces to calculating the geodesics on the product ansatz manifold.

In summary, we underline the salient points of our treatment. In principle, we begin with QCD, which implies the Skyrme model as a low energy effective field theory. The classical soliton dynamics of the Skyrme model, can be harnessed using the truncation to a finite number of relevant degrees of freedom. The ensuing dynamics for the collective coordinates reduces to geodesic motion on the manifold parametrized by the product ansatz. Our treatment requires no ad hoc parameters and each approximation has a well defined domain of validity. We are presently working out the details of projecting onto semi-classically quantized nucleonic states and the implications for nucleon-nucleon scattering.

Acknowlegements

We thank M. Temple-Raston for useful discussions. This work supported in part by NSERC of Canada and FCAR of Québec.

REFERENCES

1. N.S. Manton, *Phys. Rev. Lett.* **60**, 1916 (1988).
2. G.W. Gibbons, N. S. Manton, *Nucl. Phys.* **B274**, 183 (1986).
3. T.M. Samols, *Comm. Math. Phys.* **145**, 149 (1992).
4. T.H.R. Skyrme, *Proc. Roy. Soc. Lon.* **260**, 127 (1961).
5. E. Witten, *Nucl. Phys.* **B223**, 422, 433 (1983).
6. M. Kutschera, C.J. Pethick, *Nucl. Phys.* **A440**, 670 (1985).
7. K. Isler, J. LeTourneux and M.B. Paranjape, *Phys. Rev. D* **43**, 1366 (1991), and references herein.
8. T.S. Walhout and J. Wambach, *Phys. Rev. Lett.* **67**, 314 (1991).
9. B. Schroers, Durham preprint, DTP 93-29, hep-ph/9308236.
10. G.S. Adkins, C.R. Nappi and E. Witten, *Nucl. Phys.* **B228**, 552 (1983).